- Final Exam Advice
  - PRACTICE
  - Know the definitions and theorem really well
  - Be comfortable visualizing curves and surfaces
  - Be comfortable switching between coordinate systems (Cartesian, polar, cylindrical, spherical)
  - This is not about being really good at computing things; this is about understanding
    → Can you explain the concepts to yourself and to your peers?
  - BE CONFIDENT

Course Overview

Infinite Series

$$\sum_{n=0}^{\infty} a_n \qquad a_0 + a_1 + a_2 + \ldots$$

<u>Ex</u>: $\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r|<1$

Partial sum: $\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r}$ for any r.

When does taking a sum of infinitely many numbers make sense?
What does it mean for a series to converge?

We need $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ to exist.

<u>Ex</u>: · Geometric converges iff $|r|<1$.

· p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p>1$

In practice, finding the limit is hard. So we have convergence tests!

- Divergence Test
- Comparison Test
- Limit Comparison Test
- Integral Test
- Alternating Series Test
- Ratio Test
- Root Test

The convergence tests are really about the terms that we are summing. If they decay to zero "fast enough," we have convergence. If not, we have divergence.

Ex: $(-1)^n$ does not tend to zero at all
$\Rightarrow$ Diverges by divergence test

Ex: $\frac{1}{n}$ does not tend to zero fast enough

Ex: $\frac{1}{n^2}$ does tend to zero fast enough

But why do we care about series in calculus?
→ Power series! (Specifically, Taylor series).

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Power series allow us to represent complicated functions as (infinite) polynomials.

Ex: How does your calculator compute $e^{2.757}$? Using the Taylor series for e^x. How do we ensure this is accurate enough? Taylor's Theorem with remainder.

Taylor Series: For a function f having derivatives of all orders at a, its Taylor series centred at a is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$ If $a=0$, we say Maclaurin Series.

Ex: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(a=0)$ $\qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Nth Taylor Polynomial: $T_n(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Ex: For e^x, $T_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

For $\sin x$, $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

For $\cos x$, $T_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

Note:
- 1st Taylor polynomial is a linear approximation to f
- 2nd Taylor polynomial is a quadratic approximation to f
- 3rd is cubic
- etc.

Taylor's Theorem with Remainder: Define $R_N(x) = |f(x) - T_N(x)|$.
If $|f^{(N+1)}(x)| \leq M$ for all $x \in I$ (for some interval I), then

$$|R_n(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1}.$$

Ex: What is $R_5(x)$ for $f(x) = \cos x$?

Note $|f^{(n)}(x)| = |\sin x|$ or $|f^{(n)}(x)| = |\cos x|$ $\forall x$

$\Rightarrow |f^{(6)}(x)| \leq 1$

$\Rightarrow |R_5(x)| \leq \frac{1}{6!} |x|^{N+1} \quad \forall x$

Warning: Blindly applying Taylor's theorem and computing many derivatives to get a power series for your function is not always the best. There is often a better way.

Remember that we can take derivatives and integrals of power series!

Ex: Find the power series for $f(x) = \arctan x^3$ and determine the radius of convergence.

Notice $f'(x) = \frac{3x^2}{1+x^6} = 3x^2 \sum_{n=0}^{\infty} (-1)^n x^{6n}$, $\underline{-1 < x < 1}$

$$= \sum_{n=0}^{\infty} (-1)^n 3x^{6n+2}, \quad \underline{-1 < x < 1}.$$

Taking the integral, we have

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}, \quad \underline{-1 < x < 1}.$$

Notice we have $R = 1$.

<u>Ex</u>: Evaluate $f^{(9)}(0)$ for $f(x) = x^3 e^{x^2}$ by analyzing its power series representation.

How to do this? Find the coefficient of x^9 in the power series. Equate it with $\frac{f^{(9)}(0)}{9!}$, the coefficient of x^9 in the Taylor series expansion.

$$x^3 e^{x^2} = x^3 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!}$$

$$= x^3 + x^5 + \frac{x^7}{2!} + \frac{x^9}{3!} + \cdots$$

Equate $\frac{f^{(9)}(0)}{9!} = \frac{1}{3!} \Rightarrow f^{(9)}(0) = \frac{9!}{6}$.

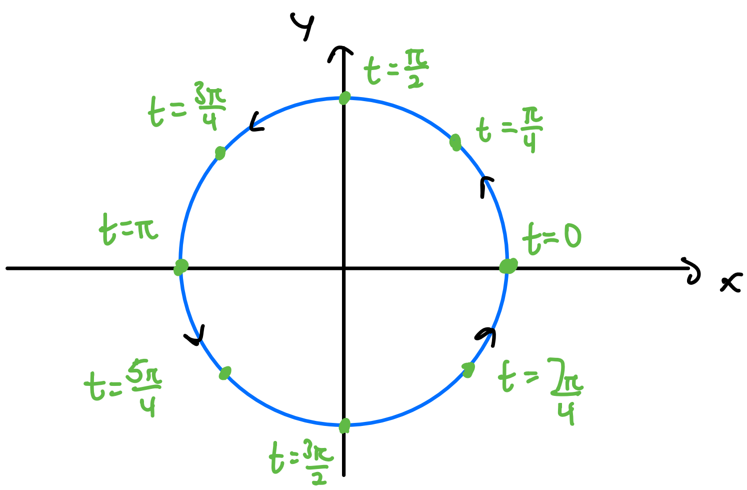
Parametric Curves and Polar Coordinates

Goal: Describe curves in $\mathbb{R}^2$. The curves are not necessarily graphs of functions and thus do not have to pass the vertical line test.

Notice the link with the later chapter on vector-valued functions, where we describe curves in $\mathbb{R}^3$!

Ex: $x = \cos t \quad y = \sin t, \quad 0 \leq t \leq 2\pi$

How to graph? Start at $t = 0$ and sample values of t along the interval.



Derivatives: Often interested in how components change with time t: $x'(t)$, $y'(t)$.

Sometimes also interested in tangent line to the curve. For this we need dy/dx.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

If $\frac{dy}{dx} = 0$, we have a <u>horizontal tangent line</u>.

If $\frac{dy}{dx} \to \pm\infty$, we have a <u>vertical tangent line</u>.

(Just like in Cal 1!)

Integrals and Area under the Curve:

$$A = \int_a^b y(t)x'(t)\,dt.$$

Intuition: For each small displacement in x (scaled by change in time), sample a y value. Take the weighted sum of such values over the interval.

Ex: See Tutorial 5 Question 2

Arc Length:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Intuition: Sum up the distances travelled over short increments of time.

Ex: See Tutorial 5 Question 3

Polar Coordinates: Many curves are easier to describe in terms of distance from the origin (r) and angle with the x-axis (θ).

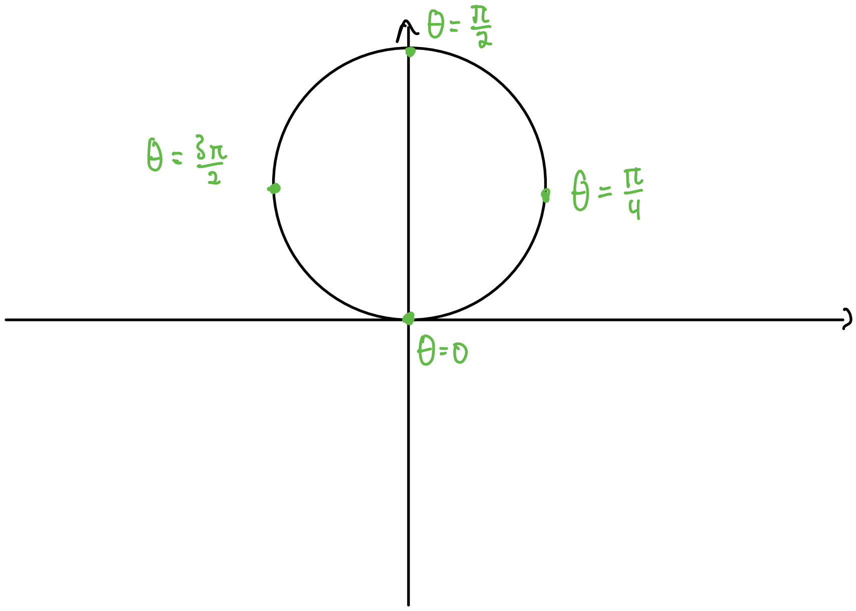
$$r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Polar functions: $r(\theta)$

How to graph these? Just like parametric curves, sample values of θ.

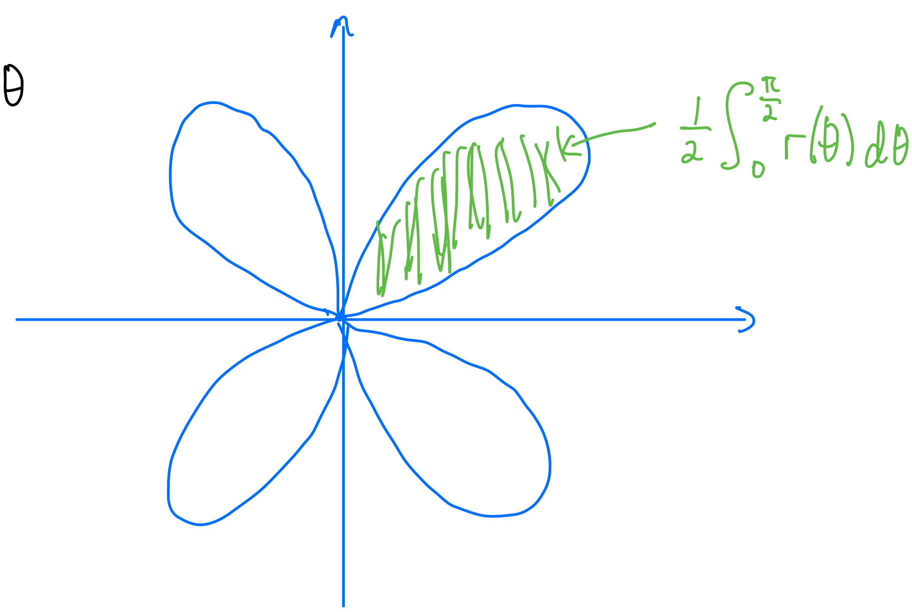
$\underline{Ex}$: $r(\theta) = 4\sin\theta$



Area of Region Enclosed by a polar curve:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r(\theta)\, d\theta$$

Intuition:



Ex: See Tutorial 5 Question 5.

Arc Length for Polar Curves:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Vectors in Space

$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

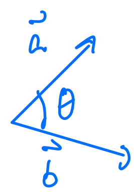
Scalar
generalizes to any dimension

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

vector
This only works in $\mathbb{R}^3$

The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$



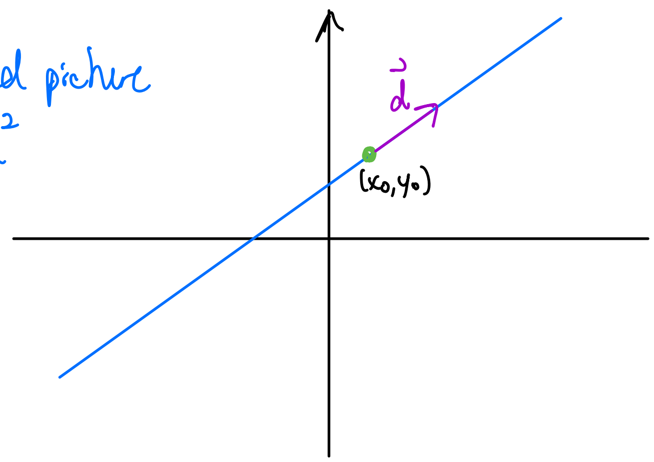
$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$

Describing lines in $\mathbb{R}^3$:

Point $\langle x_0, y_0, z_0 \rangle$ and direction vector $\vec{d} = \langle a, b, c \rangle$

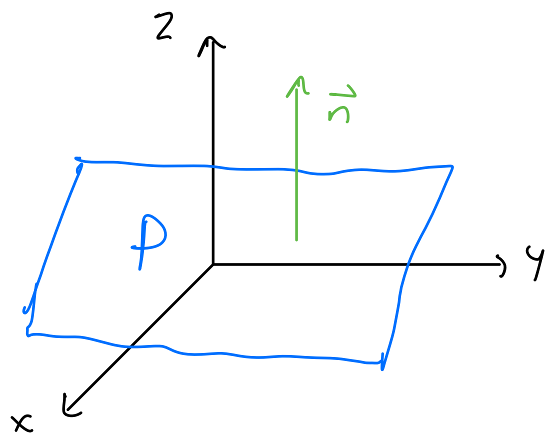
Parametric Equation of a Line: $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$, $t \in \mathbb{R}$

Simplified picture in $\mathbb{R}^2$



<u>Ex</u>: See Tutorial 6 Question 3

Describing Planes in $\mathbb{R}^3$: A plane is best understood as the set of vectors orthogonal to the normal vector $\vec{n}$.



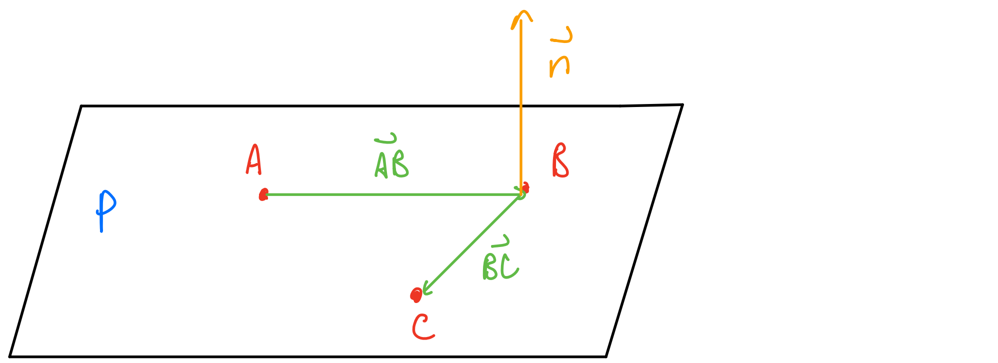
Sufficient to have a point $\langle x_0, y_0, z_0 \rangle$ and normal vector $\vec{n} = \langle a, b, c \rangle$.

Equation of a plane: $\left(\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle\right) \cdot \vec{n} = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

One can also form a plane from three points A, B, C

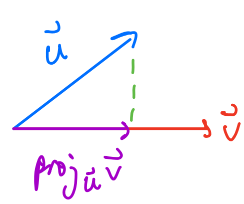
$\vec{n} = \vec{AB} \times \vec{BC}$. Why? Because the vectors from A to B and from B to C must be in the plane. Take their cross product to get a direction orthogonal to both $\vec{AB}$ and $\vec{BC}$.



Then take any of A, B, C to be $\langle x_0, y_0, z_0 \rangle$.

Ex: See Tutorial 6 Question 4

Projections: $\text{proj}_{\vec{u}} \vec{v} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\|u\|^2} \right) \vec{u}$



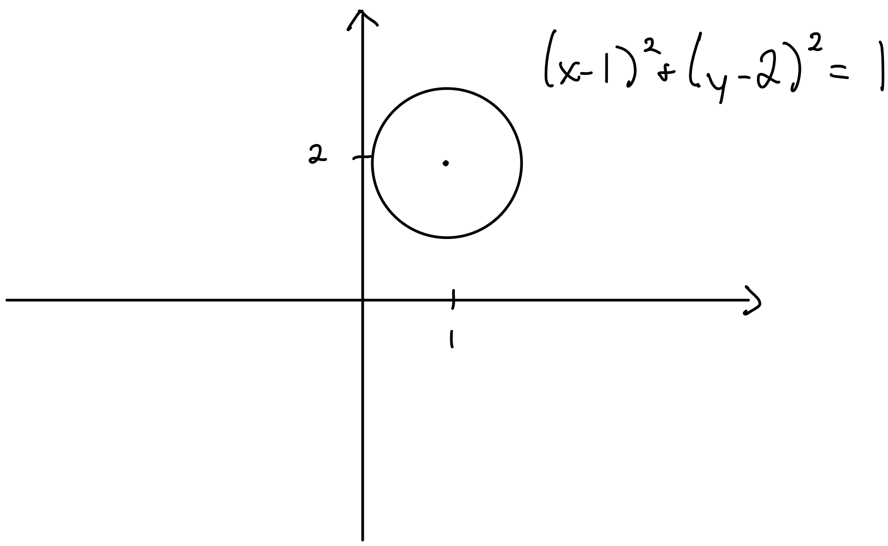
$$\text{comp}_{\vec{u}} \vec{v} = \|\text{proj}_{\vec{u}} \vec{v}\| = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$$

Note for later: Directional derivative is just norm of projection onto gradient!

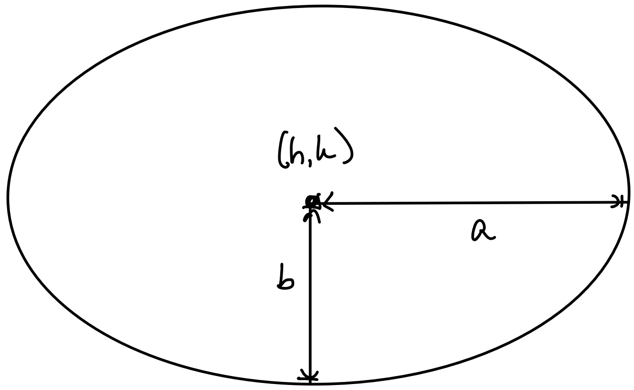
Surfaces

Know the four conic sections!

- Circle
  - Has a radius r and centre (h,k)
  - $(x-h)^2 + (y-k)^2 = r^2$



- Ellipse
  - "Generalization" of circle
  - Can think of "stretching" or "shrinking" circle along x-axis or y-axis
  - $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$



- Parabola

· $y = a(x-h)^2 + k$ or $y = ax^2 + bx + c$

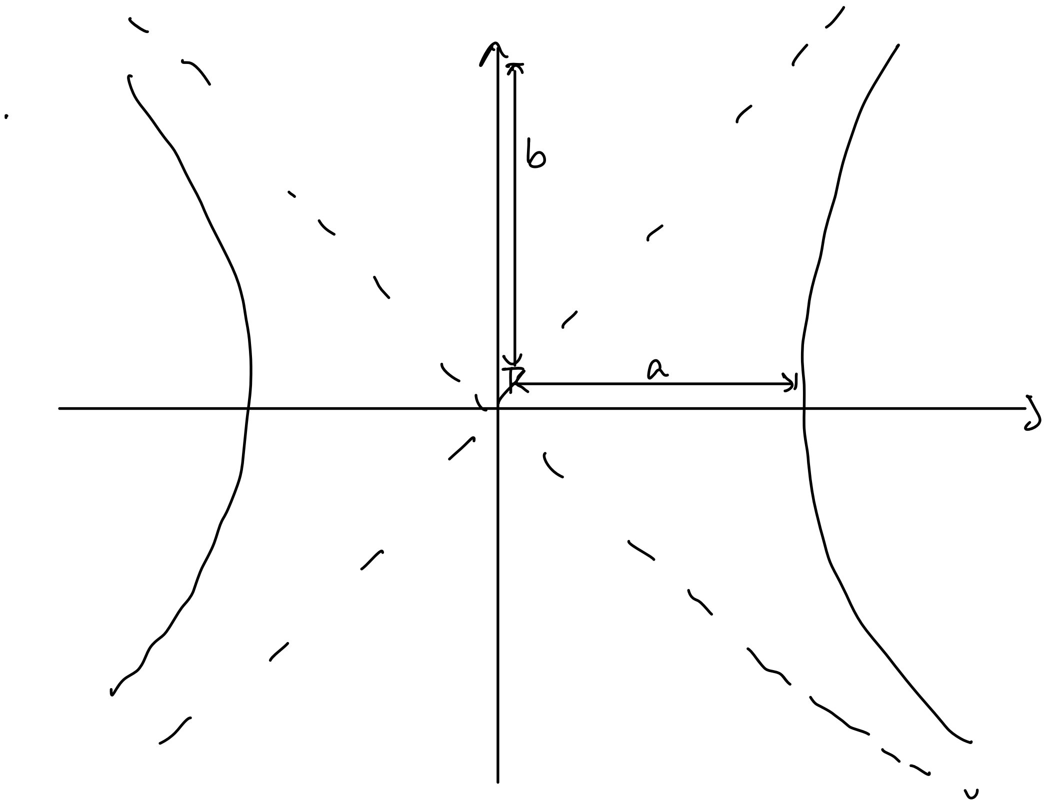
· If given second form, can find vertex by taking derivative and setting to zero

$y' = 2ax + b \qquad y' = 0 \iff x = -\frac{b}{2a}$

(Note $y'' = 2a$. If $a > 0$, concave up. If $a < 0$, concave down).

- Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



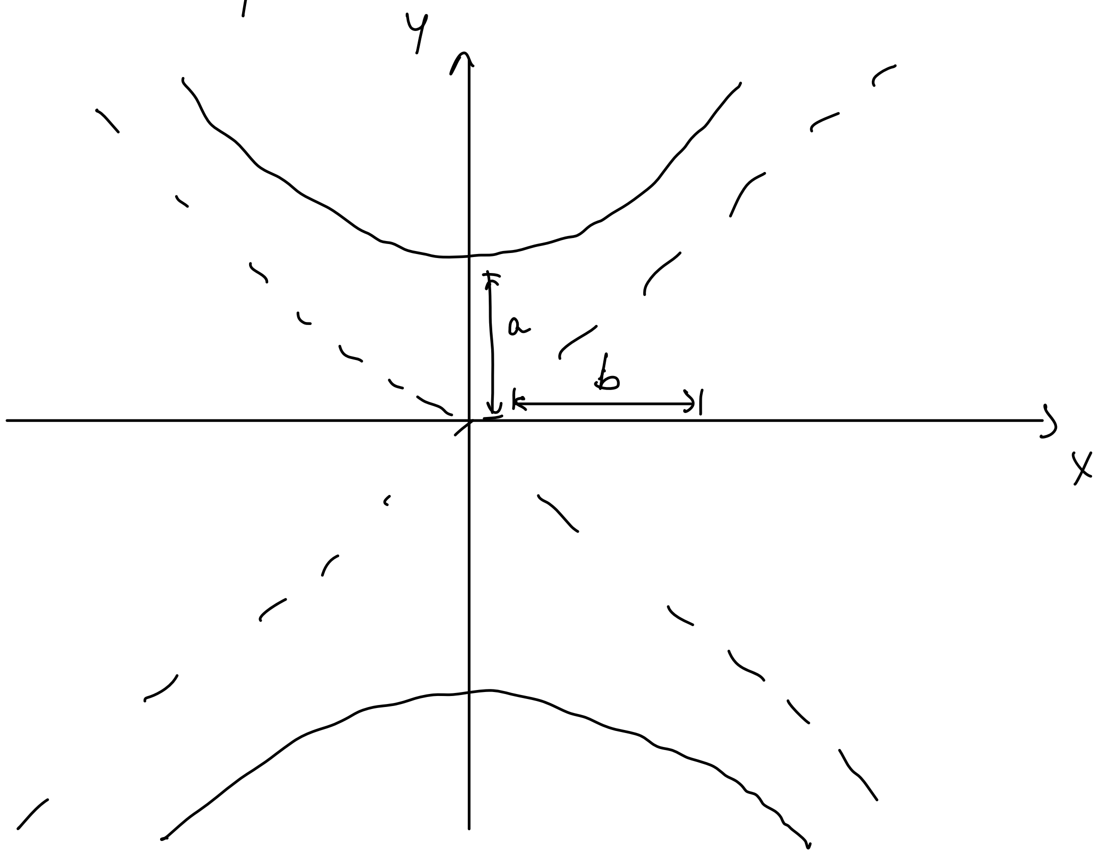
b is the "height" of the asymptote at $x = h+a$

If we assume $(h,k) = (0,0)$,

$$x^2 - y^2 = 1 \Rightarrow y^2 = x^2 - 1 \Rightarrow y = \pm\sqrt{x^2-1}$$

Helps see why we get the curve we get

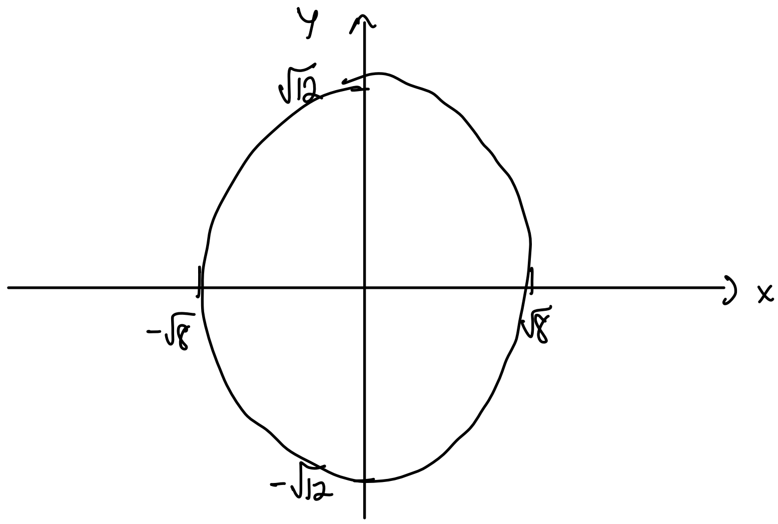
If we get the second form, imagine the same plot but flip x and y



To sketch a surface, draw the traces in the xy plane (setting z=0), xz plane (setting y=0), and yz (setting x=0).

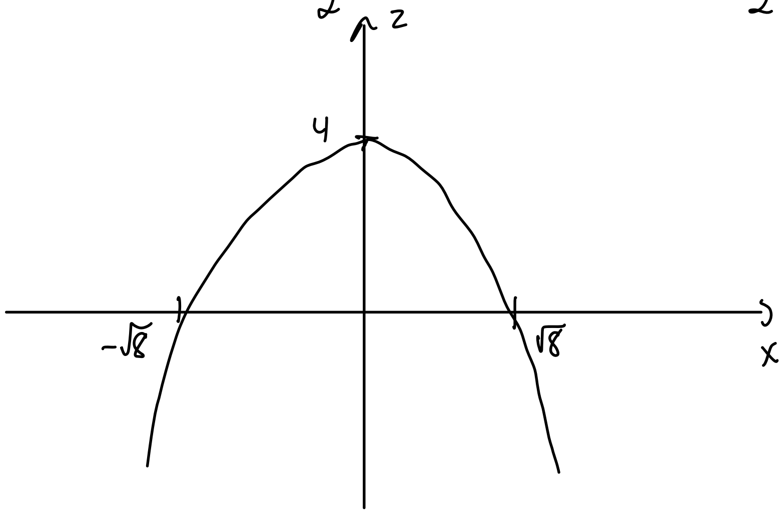
Ex: $z + \frac{x^2}{2} + \frac{y^2}{3} = 4$

xy-plane $(z=0)$: $\frac{x^2}{2} + \frac{y^2}{3} = 4 \Rightarrow \frac{x^2}{8} + \frac{y^2}{12} = 1$



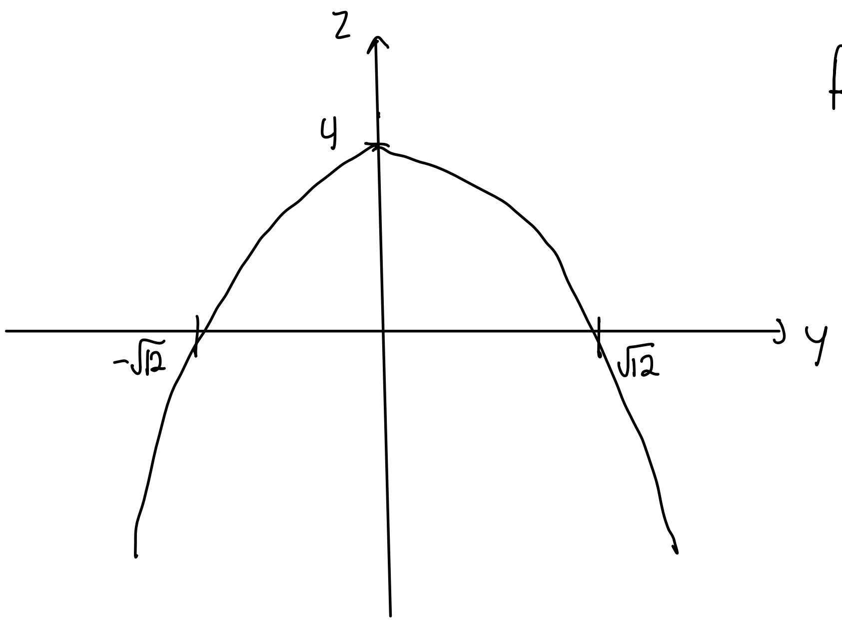
ellipse

xz-plane $(y=0)$: $z + \frac{x^2}{2} = 4 \Rightarrow z = 4 - \frac{x^2}{2}$

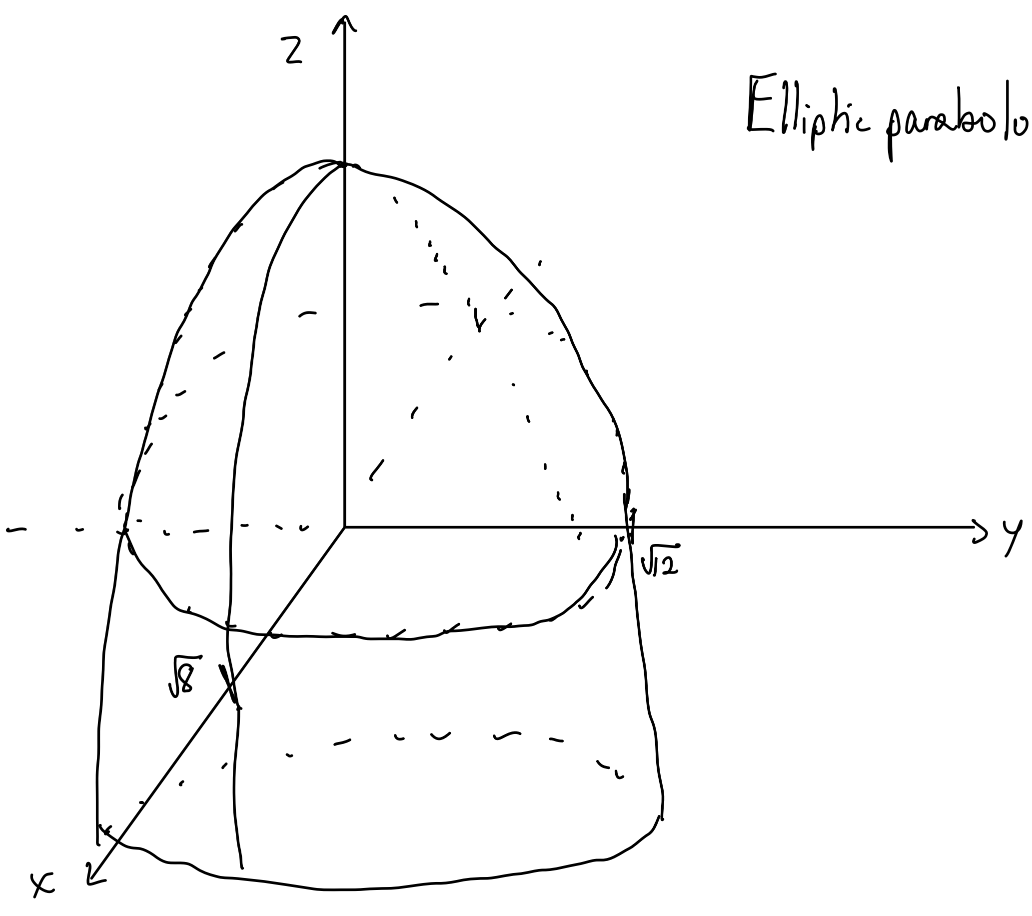


Parabola

yz plane $(x=0)$: $z + \frac{y^2}{3} = 4 \Rightarrow z = 4 - \frac{y^2}{3}$



Parabola



Elliptic paraboloid

Vector-Valued Functions

Natural extension of parametric curves to higher dimensions!

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Tangent vector: $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ provided $\|\vec{r}\,'(t)\| \neq 0$.

$\int \vec{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle$

Arc length: $s = \int_a^b \|\vec{r}\,'(t)\|\,dt$

Arc length function: If we imagine the curve as representing position over time, we can ask how much distance we have covered in a given amount of time.

$s(t) = \int_0^t \|\vec{r}\,'(t)\|\,dt.$

Ex: Find an arc-length parametrization of $\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$.

First find arc length function.

$$s(t) = \int_0^t \sqrt{16\sin^2 t + 16\cos^2 t}\ dt$$

$$= \int_0^t 4\ dt$$

$$= 4t$$

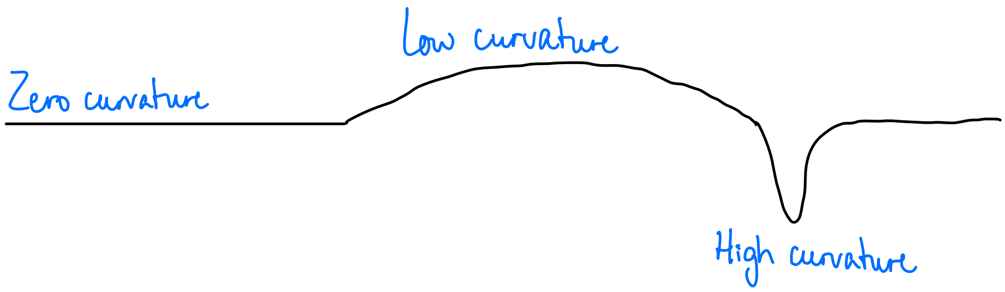
$$\Rightarrow t = \frac{s}{4}$$

We can then reparametrize the curve in terms of distance travelled (s) instead of time (t). Notice s and t have distinct meanings.

$$\vec{r}(s) = \langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right) \rangle.$$

Why would we want to think about the curve in terms of distance instead of time? It allows us to address notions like curvature!

Curvature tells us how sharply the curve is turning. This is useful for designing roads for example.



Intuition: The faster we are changing direction over a small distance, the higher the curvature. This motivates the definition:

<u>Def</u> (Curvature): The curvature of a space curve $\vec{r}(s)$, where s is the arc-length parameter, is

$$\kappa(s) = \left\| \frac{d\hat{T}}{ds} \right\| = \|\hat{T}'(s)\|.$$

In practice, it is difficult to compute curvature directly from the definition since we need the arc length parametrization. We have the following two useful alternate formulas, which we can use when given the more standard $\vec{r}(t)$ parametrization:

- $\kappa(t) = \dfrac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|}$
- $\kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

Ex: Find the curvature of $\vec{r}(t) = \langle \sqrt{2}\,t, e^t, e^{-t} \rangle$.

$\vec{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$ $\qquad \|\vec{r}'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}}$

$\vec{r}''(t) = \langle 0, e^t, e^{-t} \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{vmatrix} - \hat{j} \begin{vmatrix} \sqrt{2} & -e^{-t} \\ 0 & e^{-t} \end{vmatrix} + \hat{k} \begin{vmatrix} \sqrt{2} & e^t \\ 0 & e^t \end{vmatrix}$$

$$= (1+1)\hat{i} - \sqrt{2}e^{-t}\hat{j} + \sqrt{2}e^t\hat{k} = \langle 2, -\sqrt{2}e^{-t}, \sqrt{2}e^t \rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{4 + 2e^{-2t} + 2e^{2t}}$$

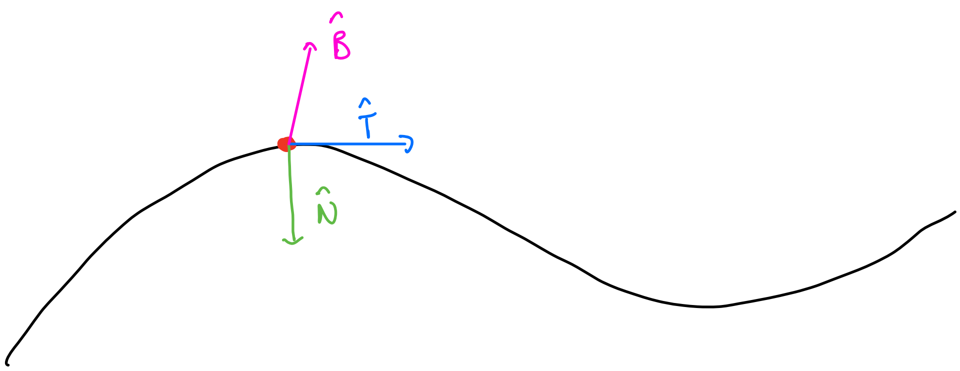
$$\kappa(t) = \frac{(4 + 2e^{-2t} + 2e^{2t})^{1/2}}{(2 + e^{2t} + e^{-2t})^{3/2}} = \frac{\sqrt{2}\,(2 + e^{-2t} + e^{2t})^{1/2}}{(2 + e^{2t} + e^{-2t})^{3/2}}$$

$$= \frac{\sqrt{2}}{2 + e^{2t} + e^{-2t}}$$

Def (Normal Vector): $\hat{N}(t) = \dfrac{\hat{T}'(t)}{\|\hat{T}(t)\|}$

Def (Binormal Vector): $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$

Note: The tangent, normal, and binormal vectors are orthogonal to each other.

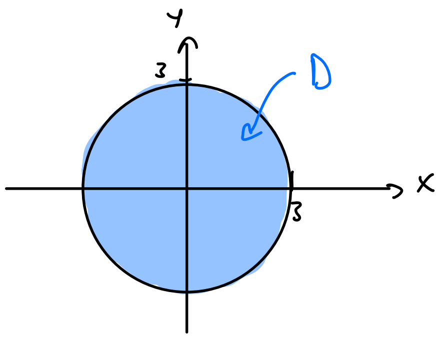


Partial Derivatives

From this point on, the course is concerned with generalizing ideas about single-variable functions (i.e. $f(x)$) to multivariable functions (i.e. $f(x,y)$, $f(x,y,z)$).

Consider a function $f(x,y)$. Its domain is a subset of $\mathbb{R}^2$ (i.e., the xy-plane) and its range is a subset of $\mathbb{R}$.

<u>Ex</u>: $f(x,y) = \sqrt{9-x^2-y^2}$ has domain $D = \{(x,y) : x^2+y^2 \leq 9\}$, i.e. the circle of radius 3 centred at the origin.



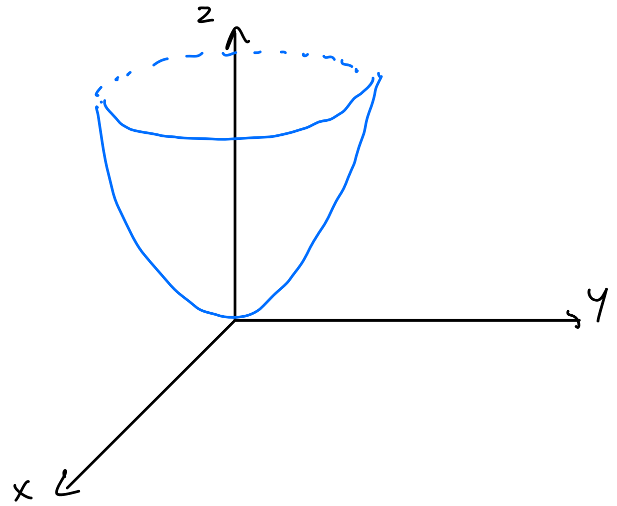
The range of f is the interval $[0,3]$.

<u>Note</u>: A clear distinction must be made between the domain and the graph of a function.

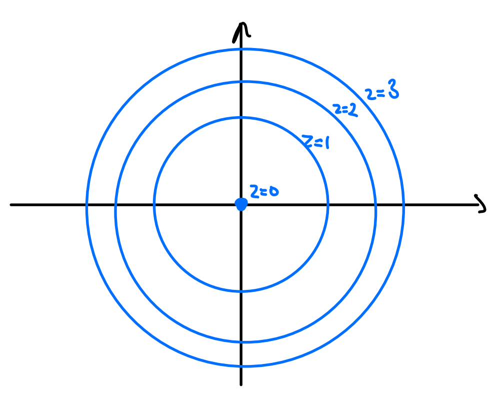
The domain of $f(x,y)$ is in $\mathbb{R}^2$ (two dimensions). The graph of $f(x,y)$ is a surface in $\mathbb{R}^3$ (three dimensions). To draw the graph of a function of two variables, set $z = f(x,y)$ to obtain the equation of a surface. Then draw the surface.

Ex: Draw the graph of $f(x,y) = x^2 + y^2$.

This boils down to drawing the surface $z = x^2 + y^2$.

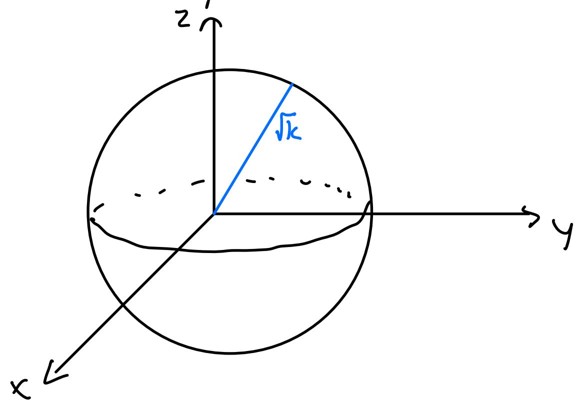


Sometimes, the 3D picture can be difficult to interpret. So, we draw level curves in the domain. $f(x,y) = k$ for different values of k.



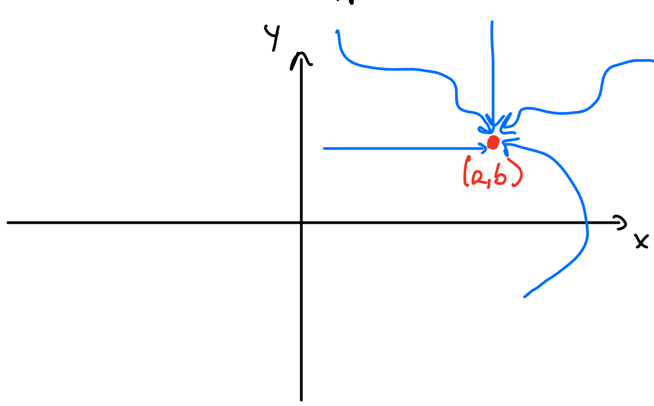
If we have a function of three variables $f(x,y,z)$, its graph is in four dimensions (we can't draw it)! In this case, we can only draw level surfaces in the 3D domain. $f(x,y,z)=k$ for different values of k.

Ex: Consider $f(x,y,z)=x^2+y^2+z^2$. The level surfaces of f are spheres of radius $\sqrt{k}$: $x^2+y^2+z^2=k$.



But now we get to the real question: how do we do calculus on these functions?

We need a notion of limit. What does it mean for $f(x,y)$ to approach L as (x,y) approaches (a,b)?



There are many ways to approach a point (a,b) in the xy-plane. In fact, there are infinitely many!

Thus, to have $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we need to have that $f(x,y)$ approaches L no matter how we approach (a,b).

If we approach from two different directions and get a different result, the limit does not exist.

<u>Ex</u>: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Try approaching along $(x,0)$ (the x-axis):

$$\lim_{(x,0)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Try approaching along the line $y=x$:

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

We get different results! Therefore, the limit does not exist.

Proving that a limit does exist is more challenging. This often requires the ε-δ method. However, you will not be asked to prove the existence of a limit via the ε-δ definition on the exam.

With the above said, there are still cases where you can compute multivariable limits, such as when the function is continuous (e.g. Tutorial 9, Q3a) or when we can reduce it to a one variable problem (e.g. Tutorial 9, Q4c).

Def (Continuity): A function $f(x,y)$ is said to be continuous at a point (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. f is called a continuous function if it is continuous at every point in its domain.

Note: Our usual continuous functions from Cal I are still continuous here.

Ex: $\sin(xy)$, x^2+y^2, $e^{-2x^3+3\sin y}$, $\ln(3x^2y)$ are all continuous.

For piecewise-defined functions, the situation gets more tricky and we have to take limits.

Ex: Determine whether the function is continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\arcsin(2x+y)}{2x+y}, & x^2+y^2<4 \text{ and } (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0). \end{cases}$$

We need to check $\lim_{(x,y)\to(0,0)} f(x,y)$. If it is 1, f is continuous at $(0,0)$. Otherwise, f is not continuous at $(0,0)$.

To simplify the limit computation, let $u=2x+y$ and notice that $u\to 0$ as $(x,y)\to(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{\arcsin(2x+y)}{2x+y} = \lim_{u\to 0} \frac{\arcsin(u)}{u}$$

$$\overset{\text{L'Hopital}}{=} \lim_{u\to 0} \frac{\frac{1}{\sqrt{1-u^2}}}{1} = 1. \quad \Rightarrow f \text{ is continuous at } (0,0)!$$

<u>Def</u> (Partial Derivatives): For a function $f(x,y)$,

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

How function changes with small change in x

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

How function changes with small change in y

All of our usual differentiation rules apply. If computing $\frac{\partial f}{\partial x}$, treat y as a constant and vice versa.

<u>Ex</u>: $f(x,y) = \sin(3xy)$

$f_x(x,y) = 3y\cos(3xy)$ $\qquad$ $f_y(x,y) = 3x\cos(3xy)$

Second-order derivatives: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$

<u>Note</u>: $\frac{\partial f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$ $\qquad$ $\frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$

<u>Thm</u>: If f_{xy} and f_{yx} are continuous, $f_{xy} = f_{yx}$.

Mixed partials are equal if they are continuous!

Def (Tangent Plane): For a function $z=f(x,y)$, if f_x and f_y exist then the tangent plane to the graph of f at (x_0,y_0) is

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Important note: The right-hand side of the above equation is an approximation of f by a linear function near (x_0,y_0). This is very useful since linear functions are often much easier to work with.

f is said to be differentiable at (x_0,y_0) if the linear approximation $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ is a "good" approximation of f near (x_0,y_0).

This definition of differentiability can be made rigorous, but we won't get into that here.

What you need to retain is this:

If f_x and f_y are continuous at (x_0,y_0) then f is differentiable at (x_0,y_0). If f is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0).

f_x and f_y continuous $\Rightarrow$ f differentiable $\Rightarrow$ f continuous

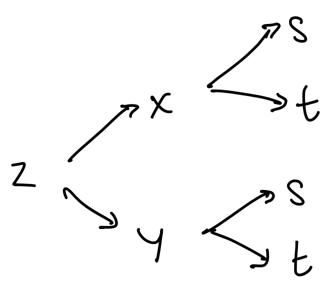
Warning: These implications do not work backwards. It is not true that all continuous functions are differentiable.

Chain Rule: There are several cases of Chain Rule. The best way to remember it (in my opinion) is via tree diagrams.

Ex: $z = f(x,y) \qquad x = x(s,t) \qquad y = y(s,t)$

$$f(x,y) = \frac{1}{(x+y)^2} \qquad x = s+t \qquad y = 2s-3t$$

Represent the dependencies as a tree:



Every arrow is a multiplication.
Every branching is an addition.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{-2}{(x+y)^3}\cdot 1 - \frac{2}{(x+y)^3}\cdot 2 = \frac{-6}{(x+y)^3}$$

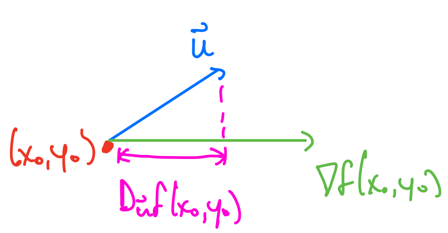
$$= \frac{-6}{((s+t)+(2s-3t))^3} = \frac{-6}{(3s-2t)^3}$$

- Directional Derivatives and the Gradient

For a function $f(x,y)$, f_x and f_y tell us how much the function changes if we move in one of the coordinate directions. But what if we move in some other direction $\vec{u}$?

Directional derivative $D_{\vec{u}} f(x_0,y_0) = \underbrace{\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle}_{\text{The gradient! } \nabla f(x_0,y_0)} \cdot \frac{\vec{u}}{\|\vec{u}\|}$

Another way of thinking about the directional derivative is as the magnitude of the projection of $\vec{u}$ onto the gradient.



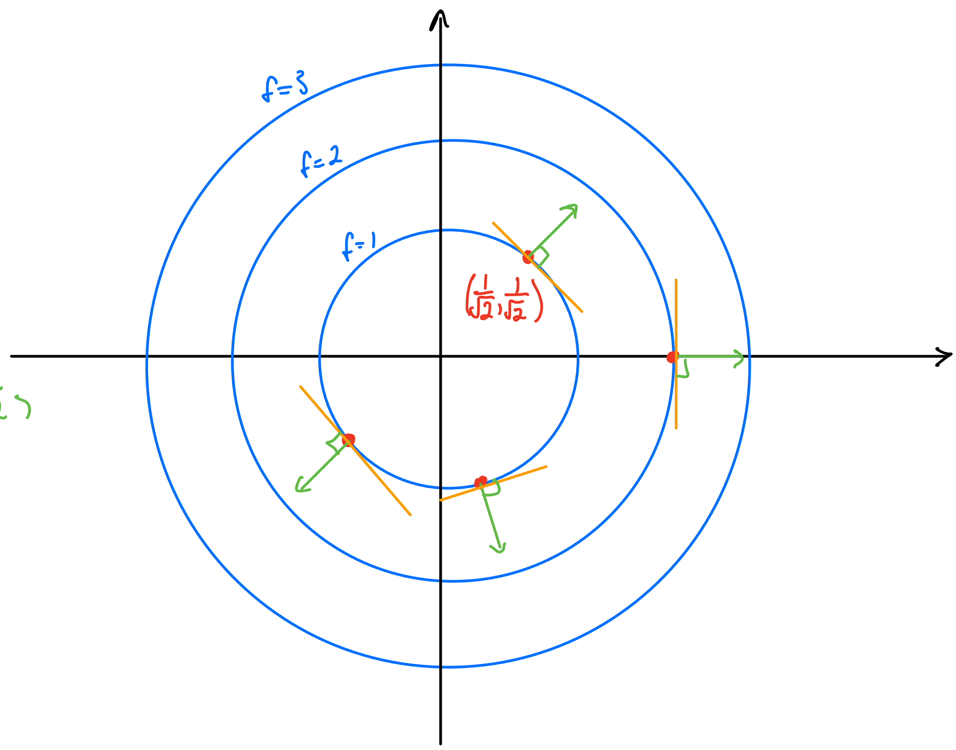
The directional derivative with the largest value is the gradient itself! The gradient indicates the "best" direction to move in to increase the function.

Let's further develop this intuition by drawing what the gradient looks like in the domain.

Ex: $f(x,y) = x^2 + y^2$

$\nabla f(x,y) = \langle 2x, 2y \rangle$

e.g. $\nabla f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \langle \sqrt{2}, \sqrt{2} \rangle$



From the figure, we see that the gradient points directly towards the "next" level curve, providing the intuition that the gradient points in the direction of steepest ascent.

Observe also that the gradient is always orthogonal to the tangent line for the level curve. Hence, the gradient provides us with a natural way to describe tangents!

The same idea applies in higher dimensions. For a function $f(x,y,z)$, the gradient $\nabla f(x_0,y_0,z_0)$ is orthogonal to the tangent plane to the level surface at (x_0,y_0,z_0).
Let's see this in action.

Ex: Find the tangent plane at $(3,2,-2)$ to the surface

$$\frac{x^2}{9} - \frac{y^2}{4} + z^2 + 5z = -6.$$

You may be tempted to try to isolate z as $f(x,y)$ and compute $z = -2 + f_x(3,2,-2)(x-3) + f_y(3,2,-2)(y-2)$. NO. That will not work here.

Instead, notice we can write the surface as the level surface $F(x,y,z) = -6$ for $F(x,y,z) = \frac{x^2}{9} - \frac{y^2}{4} + z^2 + 5z$.

Then, $\nabla F(3,2,-2)$ is the normal vector for the tangent plane we're looking for!

$$\nabla F(x,y,z) = \left\langle \frac{2x}{9}, -\frac{y}{2}, 2z+5 \right\rangle \qquad \nabla F(3,2,-2) = \left\langle \frac{2}{9}, -1, 1 \right\rangle$$

The tangent plane equation is then

$$\frac{2}{9}(x-3)-(y-2)+(z+2)=0.$$

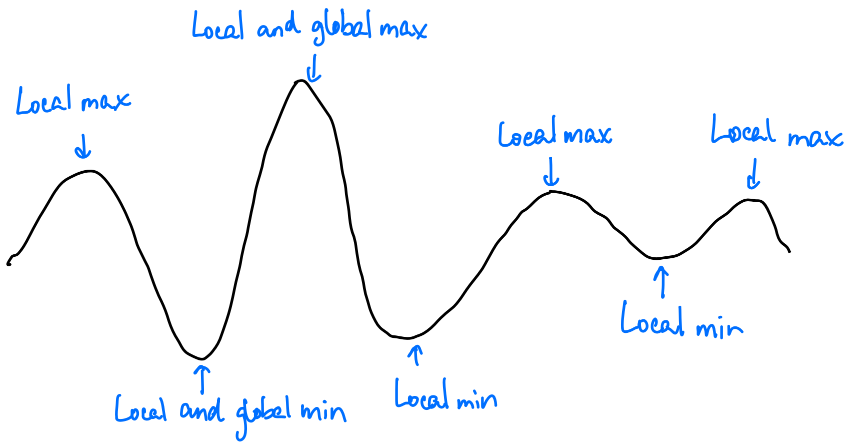
- Optimization

<u>Def</u>: Given a function $f(x,y)$ on a domain D, the point (x_0,y_0) is:

- A local minimum of f if $f(x,y) \geq f(x_0,y_0)$ on some disk centred at (x_0,y_0);
- A local maximum of f if $f(x,y) \leq f(x_0,y_0)$ on some disk centred at (x_0,y_0);
- A global minimum of f if $f(x,y) \geq f(x_0,y_0)$ for all (x,y) in D.
- A global maximum of f if $f(x,y) \leq f(x_0,y_0)$ for all (x,y) in D.

Note: A global max/min is always a local max/min, but the reverse is not necessarily true.

Intuition from 1D:



For <u>critical points</u> (i.e. $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$), we can determine whether we have a local max, min, or saddle point.

Second Derivative Test: $D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2$

- If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, <u>local min</u>;
- If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, <u>local max</u>;
- If $D < 0$, <u>saddle point</u>;
- If $D = 0$, the test is inconclusive. (x_0,y_0) may be a local min, local max, or saddle point.

<u>Ex</u>: Tutorial 11, Q5b

If the domain D is closed and bounded, we can always find an absolute maximum and an absolute minimum.

In this case you have two steps:

Step 1: Find critical points

Step 2: Check the boundary

Compile a list of points from these two steps. The point(s) which produces the smallest function value is an absolute minimum. The point(s) which produces the largest function value is an absolute maximum.

Ex: Example 4.40 in the textbook

Ex: Tutorial 11, Q4

- Lagrange Multipliers

Use this technique when you are trying to optimize a function subject to equality constraints.

Say we want to optimize $f(x,y,z)$ subject to $g(x,y,z)=0$.

Solve the system of equations
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = 0$$

After breaking up $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ into its components, we end up with four equations in four variables. List all of the solutions you get. The one producing the lowest value of f is an absolute min. The one producing the highest value of f is an absolute max.

<u>Ex</u>: Tutorial 12 Q1

<u>Ex</u>: Webwork Assignment 5 Problem 8 (Prism inside ellipsoid)

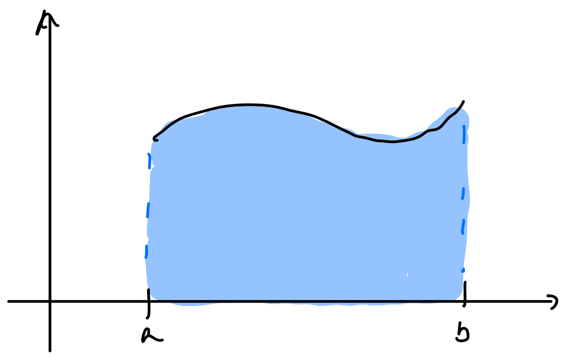
If there are two constraints $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$, solve the system

$$\nabla f(x,y,z) = \lambda_1 \nabla g(x,y,z) + \lambda_2 g(x,y,z)$$
$$g_1(x,y,z) = 0$$
$$g_2(x,y,z) = 0.$$

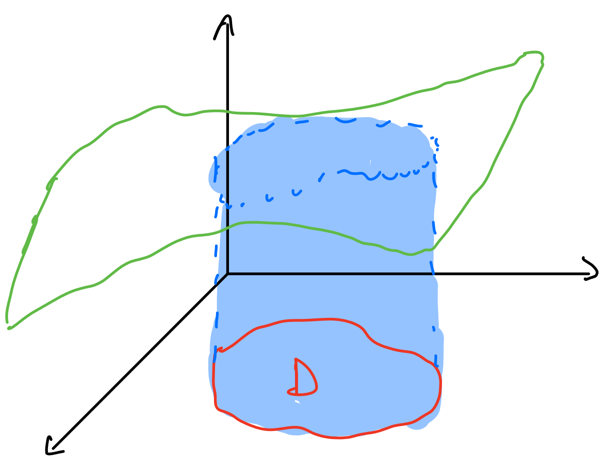
<u>Ex</u>: Tutorial 12 Q2

Multiple Integrals

In Calculus 2, we dealt with functions of one variable, so the graph of a function was in 2D. Thus, the integral represented the area under the curve for some interval.



Now, with a function of two variables, the graph of the function is in 3D. Integrals allow us to compute the volume under the surface and above some domain in the xy-plane.



Intuition: Chop up the domain of integration D into small rectangles. These form the "area of the base". The function values act as the "height".

We get $\iint_D f \, dA$ defined as the limit of a double sum

$$\iint_D f \, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \, \Delta A_{ij}.$$

For rectangular domains $D = \{(x,y) : a \le x \le b, \ c \le y \le d\}$, Fubini's Theorem gives

$$\iint_D f \, dA = \int_a^b \int_c^d f \, dy \, dx = \int_c^d \int_a^b f \, dx \, dy.$$

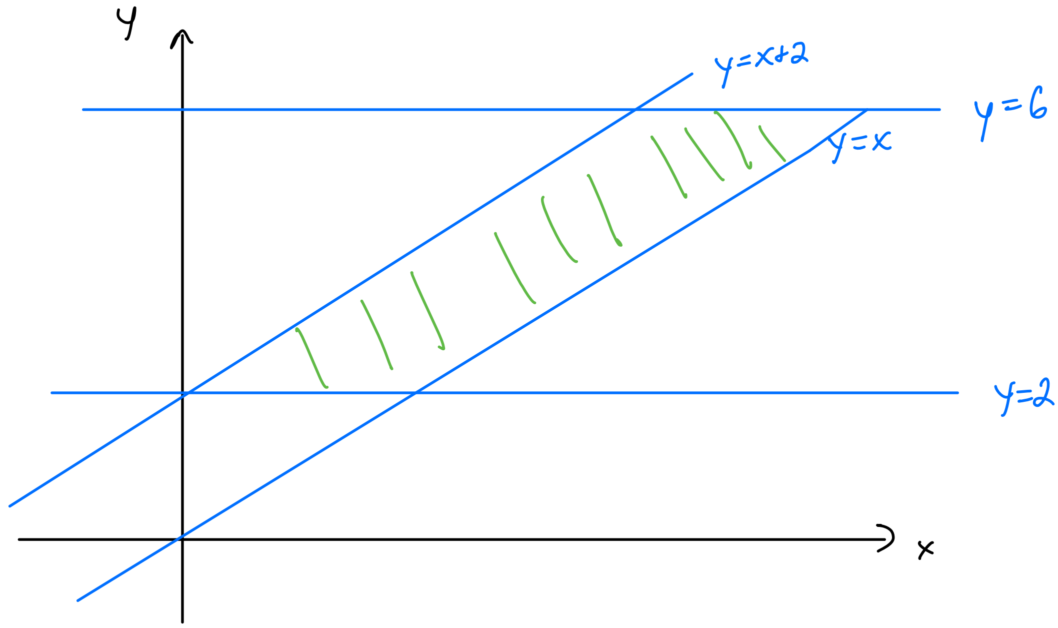
The main difficulty in multiple integrals comes from setting up integrals for non-rectangular regions.

For double integrals, there are three approaches to handle this:

- Express D as Type I: $a \le x \le b, \ g_1(x) \le y \le g_2(x)$
- Express D as Type II: $c \le y \le d, \ h_1(y) \le x \le h_2(y)$
- Express D in polar coordinates: $\iint_D f(x,y) \, dA = \iint_D f(x(r,\theta), y(r,\theta)) \, r \, dr \, d\theta$

To determine which approach to use, it is STRONGLY recommended to draw D.

Ex: $\iint_D x^2+y \, dA$ D enclosed by $y=x$, $y=x+2$, $y=2$, $y=6$



Can describe D as Type II:

$2 \le y \le 6 \qquad y-2 \le x \le y$

We get $\int_2^6 \int_{y-2}^{y} x^2+y \, dx \, dy$

$= \int_2^6 \left(\frac{x^3}{3} + xy \Big|_{x=y-2}^{x=y} \right) dy = \int_2^6 \frac{y^3}{3} + y^2 - \frac{(y-2)^3}{3} - (y-2)y \, dy$

$$= \int_2^6 \frac{y^3}{3} + \cancel{y^2} - \frac{(y-2)^3}{3} - \cancel{y^2} + 2y \; dy$$

$$= \int_2^6 \frac{y^3}{3} - \frac{(y-2)^3}{3} + 2y \; dy$$

$$= \frac{y^4}{12} - \frac{(y-2)^4}{12} + y^2 \Big|_2^6$$

$$= \frac{6^4}{12} - \frac{4^4}{12} + 36 - \frac{16}{12} - 4$$

$$= \frac{352}{3}$$

Ex: $\iint_D x^2+y^2 \, dA \qquad D = \{(x,y) : x^2+y^2 \leq 1\}$

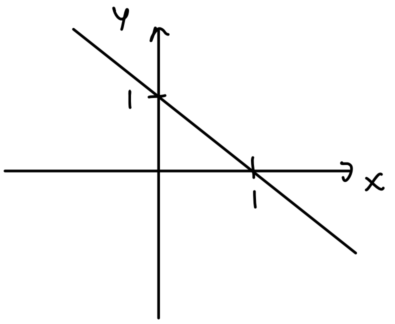
The domain is just the unit circle! Use polar coordinates.

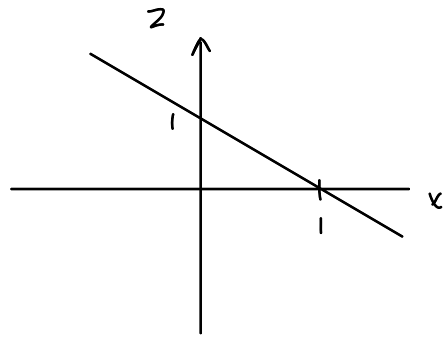
$0 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi$

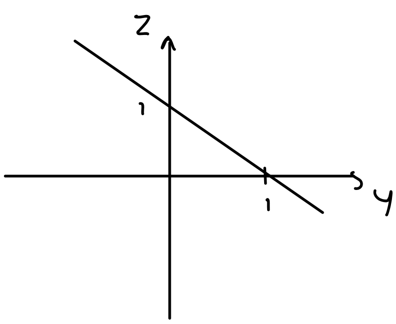
$$\int_0^{2\pi} \int_0^1 (r^2\cos^2\theta + r^2\sin^2\theta) r \, dr \, d\theta$$

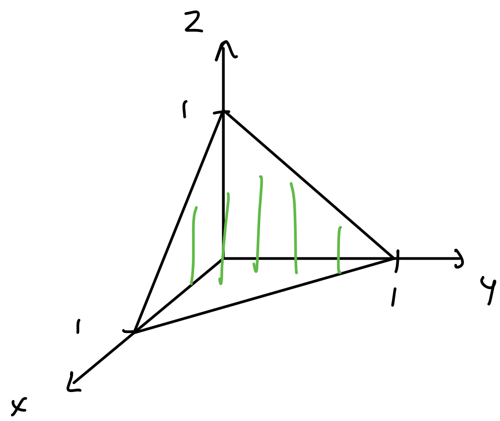
$$= \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4} \Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}$$

Ex: Volume of solid enclosed by three coordinate planes and $x+y+z=1$









$z = f(x,y) = 1-x-y \qquad D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$V = \int_0^1 \int_0^{1-x} 1-x-y \, dy \, dx$$

$$= \int_0^1 y - xy - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} dx = \int_0^1 (1-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 (1-x)^2 - \frac{(1-x)^2}{2} \, dx = \int_0^1 \frac{(1-x)^2}{2} dx$$

$$= -\frac{(1-x)^3}{6}\Bigg|_{x=0}^{x=1} = \frac{1}{6}$$

Note: If asked to compute the area of a region R in 2D, one option is to compute $\iint_R 1 \, dA$.

For triple integrals, the idea is the same as double integrals. The only thing to be cautious about is that the domain is more challenging to visualize.

Advice: Based on the constraints you're given for D, choose the variable with the most "obvious" bounds. You can then fix this variable and look at traces to determine the relationship between the other two variables.

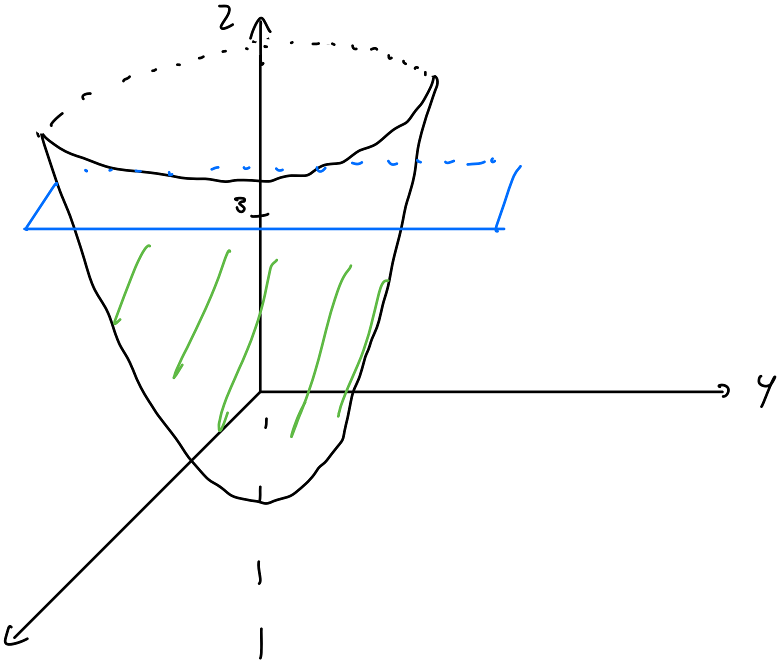
Cylindrical coordinates: Useful when it is easy to put bounds on z and to express the traces $z=k$ in polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\iiint_D f(x,y,z)\, dV = \iiint_D f(x(r,\theta), y(r,\theta), z)\, dz\, dr\, d\theta$$

Ex: $\iiint_D 3xy\, dV$

D bounded by $z = 4x^2+4y^2-1$ $\quad z=3$



If we switch to cylindrical, can write $4r^2-1 \le z \le 3$

What about the bounds on r and θ?

Consider $z=3$ plane: $3 = 4x^2+4y^2-1 \Rightarrow x^2+y^2=1 \Rightarrow 0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^1\int_{4r^2-1}^{3} 3r^2\cos\theta\sin\theta\, r\, dz\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1\int_{4r^2-1}^{3} 3r^2\cos\theta\sin\theta\, r\, dz\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 2r^3\cos\theta\sin\theta \Big|_{z=4r^2-1}^{z=3} dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 4r^3\cos\theta\sin\theta - 4r^5\cos\theta\sin\theta\, dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 (\cos\theta\sin\theta)(4r^3 - 4r^5)\, dr\,d\theta$$

$$= 12\int_0^{2\pi} (\cos\theta\sin\theta)\left(\frac{r^4}{4} - \frac{r^6}{6}\right)\Big|_{r=0}^{r=1} d\theta$$

$$= 12\int_0^{2\pi} (\cos\theta\sin\theta)\left(\frac{1}{4} - \frac{1}{6}\right) d\theta$$

$$= \cancel{12}\int_0^{2\pi} (\cos\theta\sin\theta)\,\frac{1}{\cancel{12}}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\sin 2\theta\, d\theta$$

$$= -\frac{1}{4}\cos 2\theta \Big|_{\theta=0}^{\theta=2\pi} = 0$$

Spherical coordinates: Useful when it is easy to express distance from the origin.

$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$

$$\iiint_D f(x,y,z)\,dV = \iiint_D f(x(\rho,\theta,\varphi), y(\rho,\theta,\varphi), z(\rho,\theta,\varphi))\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

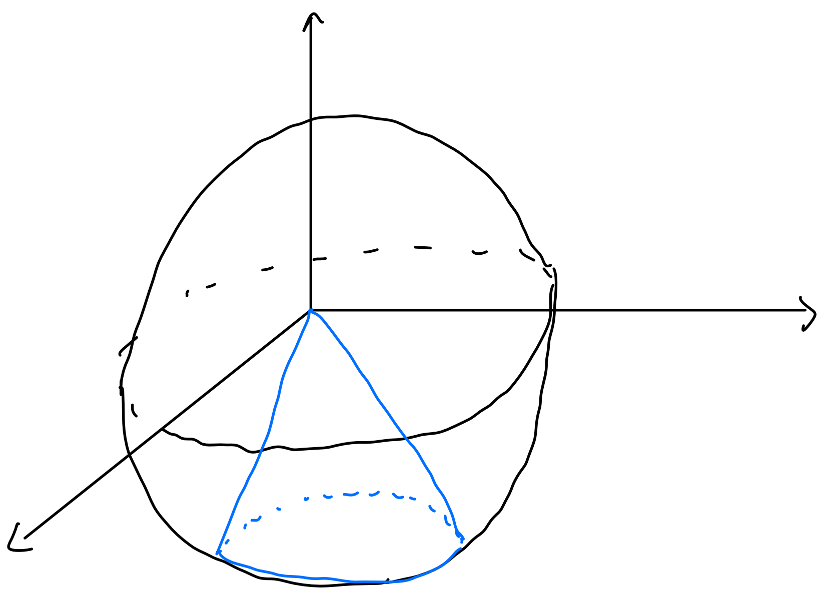
Note: The maximum range of φ is 0 to π.

The maximum range of θ is 0 to 2π.

Why? Because if we allowed $0 \le \varphi \le 2\pi$, we would be able to express certain points in multiple ways. We don't want this.

Ex: We would be able to express $(0,-1,0)$ as $\rho = 1, \theta = \frac{3\pi}{2}, \varphi = \frac{\pi}{2}$ or as $\rho = 1, \theta = \frac{\pi}{2}, \varphi = \frac{3\pi}{2}$. By restricting φ to $[0,\pi]$, we ensure the second expression is invalid.

Ex: Volume of D enclosed by sphere $x^2+y^2+z^2=4$ and cone $z=-\sqrt{x^2+y^2}$



Sphere $\rho^2 = 4$ Take $0 \le \rho \le 2$

From the cone, $\frac{3\pi}{4} \le \varphi \le \pi$

Since cross-sections are circles, $0 \le \theta \le 2\pi$

To see the cone,

$$z = -\sqrt{x^2+y^2}$$

$$\rho\cos\varphi = -\rho\sin\varphi$$

$$\Rightarrow \cos\varphi = -\sin\varphi \qquad \Rightarrow \varphi = \frac{3\pi}{4}$$

$$V = \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi} \frac{\rho^3}{3} \sin\varphi \Big|_{\rho=0}^{\rho=2} d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi} \frac{8}{3} \sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3} \cos\varphi \Big|_{\varphi=\frac{3\pi}{4}}^{\varphi=\pi} d\theta$$

$$= \int_0^{2\pi} \frac{8}{3} - \frac{8\sqrt{2}}{6} \, d\theta$$

$$= 2\pi\left(\frac{8}{3} - \frac{4\sqrt{2}}{3}\right) = \pi\left(\frac{16 - 8\sqrt{2}}{3}\right)$$